GT 2

2020/11/18

Jiao

P1

Questions précédents:

T^* 1er blowup

① T_{max} : le temps maximal d'existence. $L^\infty_t L^3_x$ espace critique

Leray (34'): T_{max}, sol regulière. Alors pour $q > 3$

$$\forall t \in (0,T), \quad \|u(\cdot,t)\|_{L^q} \geq \frac{C_p}{(\sqrt{T^*-t})^{1-\frac{3}{q}}}.$$

C_p depend de p, non $u(x,t)$

quantitatif

Espace non critique $\rightsquigarrow$ uniformément

Non critique

Seregin (14'): Non telle uniformée fonction.

Soit $\exists f$ t.q $f(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

et $\|u(\cdot,t)\|_{L^3} \gtrsim f(T^* - t)$

En effet, pour $t=0$, $\|u(\cdot,0)\|_{L^3} \gtrsim f(T^*)$

$u^\lambda(y,s) = \lambda u(\lambda y, \lambda^2 s)$

$$\Rightarrow \|u(\cdot,0)\|_{L^3} = \|u^\lambda(\cdot,0)\|_{L^3} \gtrsim f\left(\frac{T^*}{\lambda^2}\right) \xrightarrow[\lambda\to\infty]{} \infty$$

contradiction !

Tao (19')

Supposons T^* 1er temps blow-up. Alors. $\displaystyle\lim_{t\to T^*} \frac{\|u\|_{L^3(\mathbb{R}^3)}}{\left(\log\log\log\left(\frac{C}{T^*-t}\right)\right)^c} = \infty$

$$\Rightarrow \lim_{t\to T^*} \|u(\cdot,t)\|_{L^3} \gtrsim \underbrace{C}_{\text{dépend de } u} \log\log\log\left(\frac{C}{T^*-t}\right), \quad \text{Non uniformément.}$$

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ex: $f_n(t) = n^{-2}t$. $\displaystyle\lim_{t\to\infty} \frac{f_n(t)}{t^{\frac12}} = +\infty \not\Rightarrow \lim_{t\to\infty} f_n(t) \underset{\text{non uniforme}}{\gtrsim} t^{\frac12} \quad \underline{\forall n}$

✓

P2

② : blow-up :

The first instant of time T when singularities occur is called a *blow up* time. By definition, $z_0 = (x_0, t_0)$ is a *regular* point of v if it is essentially bounded in a nonempty parabolic ball with the center at the point z_0.[1] The point z_0 is *singular* if it is not regular.

③ : Différence :

ESS 03'

$\limsup_{t \to T^*} \|u(\cdot, t)\|_{L^3} = \infty$

Seregin 12'

$\underbrace{\lim_{t \to T^*} \|u(\cdot, t)\|_{L^3} = \infty}_{\text{plus forte}}$

$\limsup (-1)^n = 1$

$\lim (-1)^n$ x

✓

P3

Voir livre de PG 02' ou l'article de Jia & Sverak 14'

Definition 3.1 (Leray solution) A vector field $u \in L^2_{loc}(R^3 \times [0, \infty))$ is called a Leray solution to Navier-Stokes equations ~~with initial~~ data u_0 if it satisfies:

(i) $\operatorname{ess\,sup}_{0 \le t < R^2} \sup_{x_0 \in R^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x,t)dx + \sup_{x_0 \in R^3} \int_0^{R^2} \times \int_{B_R(x_0)} |\nabla u|^2 dx dt < \infty$, and

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2(x,t) dx dt = 0,$$

decay condition

for any $R < \infty$.

(ii) for some distribution p in $R^3 \times (0, \infty)$, (u, p) verifies Navier Stokes equations

$$\left.\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } R^3 \times (0, \infty), \tag{3.1}$$

in the sense of distributions and for any compact set $K \subseteq R^3$, $\lim_{t \to 0+} \|u(\cdot, t) - u_0\|_{L^2(K)} = 0$.

(iii) u is suitable in the sense of Caffarelli-Kohn-Nirenberg, more precisely, the following local energy inequality holds:

général que

$$\int_0^\infty \int_{R^3} |\nabla u|^2 \phi(x,t) dx dt \le \int_0^\infty \int_{R^3} \frac{|u|^2}{2}(\partial_t \phi + \Delta \phi) + \frac{|u|^2}{2} u \cdot \nabla \phi + pu \cdot \nabla \phi dx dt \tag{3.2}$$

for any smooth $\phi \ge 0$ with $\operatorname{supp} \phi \Subset R^3 \times (0, \infty)$. The set of all Leray solutions starting from u_0 will be denoted as $\mathcal{N}(u_0)$.

Estimation à priori:

$$\operatorname{ess\,sup}_{0 \le t \le R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x,t)\,dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla u|^2(x,t)\,dx\,dt \le C \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u_0|^2}{2}(x)\,dx < \infty$$

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |p - p_{x_0}(t)|^{3/2}\,dx\,dt \le C R^{1/2} \left(\sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u_0|^2}{2}(x)\,dx \right)^{3/2}$$

sol Leray-Hopf sol d'energie finie

bd $\quad L^\infty L^2 \cap L^2 H^1$ $\quad$ es... globale

P4

page 255. (J&S 14')

Theorem 3.2 (Local Hölder regularity of Leray solutions) *Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u|^2(x)dx \le \alpha < \infty$. Suppose u_0 is in $C^\gamma(B_2(0))$ with $\|u_0\|_{C^\gamma(B_2(0))} \le M < \infty$. Then there exists a positive $T = T(\alpha, \gamma, M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies:*

$$u \in C^\gamma_{\text{par}}(\overline{B_{1/4}} \times [0, T]), \quad \text{and} \quad \|u\|_{C^\gamma_{\text{par}}(\overline{B_{1/4}} \times [0,T])} \le C(M, \alpha, \gamma). \quad (3.12)$$

la method:

$u = a + V$ a est une sol mild

V est une perturbation.

étape 1 : Estimations d'énergie locale pour V

étape 2 : ε-régularité

Eq-perturbation:

$$\partial_t V - \Delta V + a\cdot\nabla V + \operatorname{div}(a \otimes V) + V\cdot\nabla V + \nabla q = 0$$

① $\partial_t u - \Delta u + u\cdot\nabla u + \nabla P = 0$

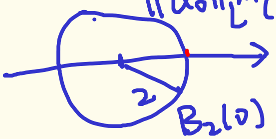
② $\partial_t a - \Delta a + a\cdot\nabla a + \nabla \tilde{p} = 0$ ① − ② $\nearrow$ $q = P - \tilde{p}$

étap 1 **Theorem 3.1** *Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u_0|^2(x)dx \le \alpha < \infty$. Suppose u_0 is in $L^m(B_2(0))$ with $\|u_0\|_{L^m(B_2(0))} \le M < \infty$ and $m > 3$. Let us decompose*[1] *$u_0 = u_0^1 + u_0^2$ with $\operatorname{div} u_0^1 = 0$, $u_0^1|_{B_{4/3}} = u_0$, $\operatorname{supp} u_0^1 \Subset B_2(0)$ and $\|u_0^1\|_{L^m(R^3)} \le C(M, m)$. Let a be the locally in time defined mild solution to Navier-Stokes equations with initial data u_0^1. Then there exists a positive $T = T(\alpha, m, M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies: $u - a \in C^\gamma_{\text{par}}(\overline{B_{1/2}} \times [0, T])$, and $\|u - a\|_{C^\gamma_{\text{par}}(\overline{B_{1/2}} \times [0,T])} \le C(M, m, \alpha)$, for some $\gamma = \gamma(m) \in (0, 1)$.*

$\|u_0^1\|_{L^m(\mathbb{R}^3)} \le C$

$\Rightarrow \|u_0^1\|_{L^m(B_2(0))} \le C$

$\|u_0\|_{L^m(B_2(0))} \le M < \infty$



$u_0^1|_{4/3} = u_0$

P5

ε-régularité :

$$\begin{cases} \partial_t V - \Delta V + a\cdot\nabla V + \operatorname{div}(a\otimes V) + V\cdot\nabla V + \nabla q = 0 \\ \operatorname{div} V = 0 \end{cases}$$

Theorem 2.2 (Improved ϵ-regularity criteria) *Let* (V, q) *be a suitable weak solution to Eq.* (2.1) *in* Q_1*, with* $a \in L^m(Q_1)$, $\operatorname{div} a = 0$, $\|a\|_{L^m(Q_1)} \leq M$*, for some* $M > 0$ *and* $m > 5$*. Then there exists* $\epsilon_1 = \epsilon_1(m, M) > 0$ *with the following properties: if*

$$\left(\fint_{Q_1} |V|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3} \leq \epsilon_1,$$

then V *is Hölder continuous in* $Q_{1/2}$ *with exponent* $\alpha = \alpha(m) > 0$ *and*

$$\|V\|_{C^{\alpha}_{\mathrm{par}}(Q_{1/2})} \leq C(m, \epsilon_1, M) = C(m, M). \tag{2.22}$$

à faire

p6

$V = u - \underline{a}$

Pf of Thm II:

a est une sol mild de NS. avec $\underline{u_0^1}$.

$$\begin{cases} \partial_t a - \Delta a + a\cdot\nabla a + \nabla\tilde{p} = 0 \\ \operatorname{div} a = 0 \quad , \quad a(\cdot,0) = u_0^1 \in \mathbb{R}^3. \end{cases}$$

par la thy des sol mild:

$$\mathbb{P}(\partial_t a - \Delta a + a\cdot\nabla a) + \cancel{\mathbb{P}(\nabla\tilde{p})} = 0$$

$$\Rightarrow \quad \partial_t a = \Delta a + \mathbb{P}(a\cdot\nabla a)$$

$$\Rightarrow a(t,x) = e^{t\Delta}u_0^1(x) + \underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}\operatorname{div}(a\otimes a)(x,s)\,ds}_{B(a,a)}$$

$B : \underset{a}{X} \times \underset{a}{X} \to \underline{X}$. bilinear et contin

$\|B(a.a)\|_X \le C\|a\|_X\|a\|_X$

point fixe.

pour les espaces critiques:

$X_3 = \{ f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3 : \|f\|_{X_3} = \sup t^{\frac{1}{2}-\frac{3}{2q}}\|f\|_{L^q} < \infty \}$.

quand $q=3$. difficile. car $\|B(a.a)\|_{X_3}$ n'est pas borné dans $C(0,T; L^3(\mathbb{R}^3))$. Oru 98'

mais ici, on a $u_0^1 \in L^m(\mathbb{R}^3)$. $m>3$. Non critique.

Regarde l'eq de la chaleur. $\begin{cases}\partial_t h - \Delta h = 0\\ h(\cdot,0)=h_0.\end{cases}$

$\Rightarrow \quad \underline{\|h\|_{L^p_t L^q_x} \lesssim \|h_0\|_{L^r_x} \quad \text{avec :} \quad \frac{1}{p} = \frac{3}{2}\left(\frac{1}{r} - \frac{1}{q}\right)}$

Ici on a $u_0^1 \in L^m(\mathbb{R}^3)$, $\quad m>3$.

$\Rightarrow \|e^{t\Delta}u_0^1\|_{L^q_t L^q_x} \lesssim \|u_0^1\|_{L^r_x} \qquad \frac{1}{q} = \frac{3}{2}\left(\frac{1}{r} - \frac{1}{q}\right)$

$\Rightarrow \|e^{t\Delta}u_0^1\|_{L^{\frac{5m}{3}}_{t,x}} \le \|u_0^1\|_{L^m_x} \qquad q = \frac{5m}{3} > 5$

$\Rightarrow \|a\|_{L^{\frac{5m}{3}}_{t,x}} \le \|u_0^1\|_{L^m_x} \le C_{M,m} \qquad m>3$

$\Rightarrow$ Estimations pour $(a, \tilde{p})$

$$\sup_{0<t<T_1}\int_{B_1(x_0)} |a|^2(x,t)\,dx + \int_0^{T_1}\int_{B_1(x_0)} 2|\nabla\otimes a|^2(x,t)\,dx\,dt \le C\|u_0^1\|_{L^m} \le C_{M,m}. \qquad \checkmark$$

$(a,\tilde{p})$. sol mild.

$$\frac{d}{dt}\int |a|^2\varphi = 2\int \partial_t a\cdot a\,\varphi$$
$$= 2\int a\varphi\left(\Delta a - \underline{a\cdot\nabla a} + \nabla\tilde{p}\right)$$

$$\Rightarrow \frac{d}{dt}\int |a|^2 + 2\int |\nabla\otimes a|^2 = 0$$

$\int (a\varphi)\cdot(a\cdot\nabla)a = 0$

$\operatorname{div} a = 0$

$\int (a\varphi)\cdot\nabla\tilde{p} = 0$

$\Rightarrow$

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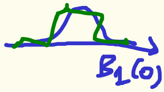
Estimation pour $\tilde{P}$.

$$\int_0^{T_1}\int_{B_1(x_0)} |\tilde{P}|^{\frac{3}{2}}\,dx\,dt \leq C_{\mu.m} \qquad \text{pour tout } x_0 \in \mathbb{R}^3.$$

$$\partial_t a - \Delta a + a\cdot\nabla a + \nabla\tilde{P} = 0 \qquad \left\| \frac{\operatorname{div}\operatorname{div}}{\Delta}\phi \right\|$$

$$\Rightarrow \Delta\tilde{P} = -\operatorname{div}\operatorname{div}(a\otimes a)$$

$$\Rightarrow \left(\int_{B_1(0)} |\tilde{P}|^{3/2}dx\right)^{\frac{2}{3}} = \|\tilde{P}\phi\|_{L^{3/2}(\mathbb{R}^3)} \quad \text{avec } \phi$$



$$= \Big\| \underbrace{-\frac{\operatorname{div}\operatorname{div}}{\Delta}}_{P} \underbrace{(a\otimes a)}_{f}\,\phi \Big\|_{L^{3/2}(\mathbb{R}^3)}$$

$$= \|(a\otimes a)P\phi - [a\otimes a, P]\phi\|_{L^{3/2}}$$

$$\leq c\|a\otimes a\|_{L^{3/2}(B_1(0))} + \text{correcteurs}$$

$$\leq c\|a\|^2_{L^3(B(0))}$$

$$\Rightarrow \int_{B_1(0)} |P|^{\frac{3}{2}} \leq c\|a\|^3_{L^3} \qquad 2\times\frac{3}{2} = 3$$

$$\Rightarrow \int_0^T\int_{B_1(0)} |P|^{3/2} \leq \int_0^T \|a\|^3_{L^3} \leq \|a\|^3_{L^3_{t,x}((0,T)\times B_1(0))} \underset{\text{Hölder}}{\leq} \|a\|_{L^{10/3}_{t,x}((0,T)\times B_1(0))}$$

$$Pf\phi = \underbrace{fP\phi}_{\text{principale}} - \underbrace{[f,P]\phi}_{\text{error terme}}.$$

On a $a \in \underset{\sim}{L^\infty L^2} \cap L^2\dot{H}^1 \Rightarrow a \in L^{10/3}_{t,x}$

$\Uparrow$

interpolation + Hölder

p9

Posons $V = u - a$, alors V vérifie l'éq au sens de distribution

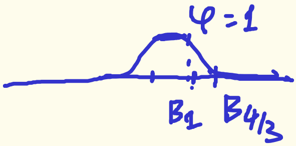
- $$\begin{cases} \partial_t V - \Delta V + a\cdot\nabla V + \underline{\mathrm{div}(a\otimes V)} + V\cdot\nabla V + \nabla q = 0 \\ \mathrm{div}\, V = 0 \end{cases}$$ Ici. $q = p - \tilde{p}$

- Et l'inégalité d'énergie locale : $V = u - a$

$$\partial_t |V|^2 - \Delta |V|^2 + 2|\nabla V|^2 + \underbrace{\mathrm{div}(|V|^2(V+a))}_{\Uparrow\ (a\cdot\nabla V + V\cdot\nabla V)} + \underbrace{2V\,\mathrm{div}(a\otimes V)}_{\Uparrow\ \mathrm{div}(a\otimes V)} + 2\,\mathrm{div}(Vq) \leq 0$$ dans $\mathbb{R}^3\times(0,T_1)$

NS : $\partial_t|u|^2 - \Delta|u|^2 + 2|\nabla u|^2 + \mathrm{div}(|u|^2 u) + 2\,\mathrm{div}(up) = 0$

Soit $\varphi \in C_0^\infty(B_{4/3}(0))$



Intégrale en x

$$\int_{\mathbb{R}^3 \to B_{4/3}} \partial_t |V|^2 \varphi(x) + 2|\nabla V|^2 \varphi(x)$$

$$\leq \int_{\mathbb{R}^3 \to B_{4/3}} \Delta|V|^2\varphi(x) - \mathrm{div}(|V|^2(V+a))\varphi(x) - 2V\,\mathrm{div}(a\otimes V)\varphi(x) - 2\,\mathrm{div}(Vq)\varphi(x)\, dx$$

$$\downarrow \int_{B_{4/3}} |V|^2 \Delta\varphi \qquad \int_{B_{4/3}} (|V|^2(V+a))\nabla\varphi \qquad \Rightarrow \int_{B_{4/3}} (Vq)\cdot\nabla\varphi$$

$$\downarrow\ 2\int (a\otimes V) : \nabla(V\varphi)$$

$$\Downarrow\ 2\int (a\otimes V) : (\nabla V\varphi + V\otimes\nabla\varphi)$$

Intégrale en t.

$$\int_{B_{4/3}} |V(x,t)|^2\varphi(x)\,dx + 2\int_0^t\int_{B_{4/3}} |\nabla V(x,s)|^2\varphi(x)\,dx\,ds$$

$$\leq \int_0^t\int_{B_{4/3}} |V|^2\Delta\varphi + (|V|^2(V+a))\nabla\varphi + 2(a\otimes V):(\nabla V\varphi + V\otimes\nabla\varphi) + 2(Vq)\cdot\nabla\varphi\, dx\,dt$$

$$+ \int_{B_{4/3}} |V(x,0)|^2 \underbrace{\varphi(x)\,dx}_{L^\infty} \qquad u_0^2$$

$$\leq C\|V(\cdot,0)\|^2_{L^2(B_{4/3}(0))} \leftarrow \lim_{t\to 0^+} \|V(\cdot,t)\|^2_{L^2(B_{4/3})} = 0$$

P10

$\overset{0 \text{ ds } B_1(x_0)}{}$

- $\lim_{t\to 0^+} \| V(\cdot,t) - u_0^2 \|_{L^2(B_1(x_0))} = 0$. pour tout $x_0 \in \mathbb{R}^3$.

$$\begin{cases} \lim_{t\to 0^+} \| u(\cdot,t) - u_0 \|_{L^2(B_1(x_0))} = 0 \\ \lim_{t\to 0^+} \| a(\cdot,t) - u_0^1 \|_{L^2(B_1(x_0))} = 0 \end{cases}$$

Estimations pour (u,p) → sol de Leray. (voir P G 02')

$$\operatorname{ess\,sup}_{0\le t\le R^2} \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} |u|^2(x,t)\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{R^2}\int_{B_R(x_0)} 2|\nabla u|^2(x,t)\,dx\,dt \le C \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} |u_0|^2(x)\,dx < \infty \quad (1)$$

$$\sup_{x_0\in\mathbb{R}^3} \int_0^{R^2}\int_{B_R(x_0)} |p - p_{x_0}(t)|^{3/2}\,dx\,dt \le C R^{1/2} \left(\sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} |u_0|^2(x)\,dx \right)^{3/2} \quad (2)$$

$R \to 1, \; x_0 \to 0$

Estimations pour $(a, \tilde{p})$

$$\operatorname{ess\,sup}_{0<t<T_1} \int_{B_1(x_0)} |a|^2(x,t)\,dx + \int_0^{T_1}\int_{B_1(x_0)} 2|\nabla a|^2(x,t)\,dx\,dt \le C_{u,m}, \quad \text{pour tout } x_0 \in \mathbb{R}^3 \quad (3)$$

$$\int_0^{T_1}\int_{B_1(x_0)} |\tilde{p}|^{\frac{3}{2}}\,dx\,dt \le C_{u.m} \quad \text{pour tout } x_0 \in \mathbb{R}^3. \quad (4)$$

$(V, q) = (u - a, \; p - \tilde{p})$

on choisit $R = 1, \; x_0 = 0$

$$\Rightarrow \sup_{0<t<T_2} \int_{B_2(0)} |\underset{u-a}{V}|^2\,dx + \int_0^{T_2}\int_{B_2(0)} |\nabla V|^2\,dx\,ds + \left(\int_0^{T_2}\int_{B_2(0)} |q|^{3/2} \right)^{2/3} \le C_{u,m,\alpha}. \quad (5)$$

$$\Rightarrow \|V\|_{L_t^\infty L_x^2((0,T_2)\times B_2(0))} + \|V\|_{L^2 H^1 \dots} + \|q\|_{L^{3/2}(\dots)} \le C.$$

P11 : $\quad v : L^\infty L^2 \cap L^2 H^1 \quad \Rightarrow \quad v \in L^{10/3}(\underbrace{0,T_2,}\times B_{4/3}(0))$

interpolation.

Inégalité d'énergie locale :

$$\underline{\int_{B_{4/3}} |v(x,t)|^2 \varphi(x)\,dx + 2\int_0^t\int_{B_{4/3}} |\nabla v(x,s)|^2 \varphi(x)\,dx\,ds} \leq C \cdot t^{?}$$

$$\leq \underbrace{\int_0^t\int_{B_{4/3}} |v|^2 \Delta\varphi} + \underbrace{(|v|^2(v+a))\nabla\varphi} + \underbrace{2(a\otimes v):(\nabla v\varphi + v\otimes\nabla\varphi)} + \underbrace{2(vq)\cdot\nabla\varphi}\,dx\,dt$$

$$\int_0^t\int_{B_{4/3}} |v\otimes v|\,\Delta\varphi \qquad \frac{3}{10}+\frac{3}{10}+\frac{2}{5}=1$$

$L^{10/3}$, $L^{10/3}$, $L^{5/2}$

$$C\,t^{\min\left(\frac{1}{10},\,\frac{2m-3}{5m}\right)}$$

$$\int_0^t \|v\|^2_{L^{10/3}}\mathbb{1}\,ds$$

$$\leq \left\|\|v\|^2_{L^{10/3}}\right\|_{L^{5/3}} \left(\int_0^t \mathbb{1}^{\frac{5}{2}}\right)^{2/5}$$

$$C\,t^{\frac{m-3}{5m}} \qquad C\,t^{1/30}$$

$$\leq \underbrace{\|v\|_{L^{10/3}L^{10/3}}}_{\leq C}\, t^{2/5}$$

$$\leq C\, t^{2/5}. \qquad \underline{\text{si } t < T_2}$$

$\text{car } \|v\|_{L^{10/3}_{t,x}(0,T_2,B)}$

$\Rightarrow$ Estimation quantitative

$$\int_{B_{4/3}} |v(x,t)|^2 \varphi(x)\,dx + 2\int_0^t\int_{B_{4/3}} |\nabla v(x,s)|^2 \varphi(x)\,dx\,ds \leq C_{\mu,m,\alpha}\, t^{\min\left(\frac{1}{30},\,\frac{m-3}{5m}\right)}, \quad t < T_2$$

P12

Estimations pour q :

$$\Delta q = -\mathrm{div}\,(v\cdot\nabla v + a\cdot\nabla v + \mathrm{div}\,(a\otimes v))$$

$$= -\mathrm{div}\,\mathrm{div}\,(\underbrace{v\otimes v}_{L^{5/3}} + v\otimes a + a\otimes v)$$

$L^{10/3}$ $L^{10/3}$ $L^{5/3}$

$\Rightarrow q \in L^{5/3}_{loc.t.x} \leftarrow$ (C-Z + cut-off)

$\Rightarrow \int_0^t \int_{B_1(0)} |q|^{3/2} \le C_{M.m.\alpha}\, t^{1/10}$, si $t < T_2$

$\int_0^t \|q\|^{3/2}_{L^{3/2}(B_1(0))} ds$ $\qquad \frac{5}{3} > \frac{3}{2}$

$\le$

$\int_0^t \underbrace{\|q\|^{3/2}_{L^{5/3}(B_1(0))}}_{L^{10/9}} \underbrace{\mathbb{1}}_{L^{10}} ds$

$\le$

$\|q\|_{L^{5/3}_t L^{5/3}_x} \left(\int_0^t 1\right)^{1/10} \le C\, t^{\frac{1}{10}}$, si $t < T_2$.

petit $< \varepsilon$

Theorem III.2 (Improved ϵ-regularity criteria) *Let (V, q) be a suitable weak solution to Eq. (2.1) in Q_1, with $a \in L^m(Q_1)$, $\mathrm{div}\, a = 0$, $\|a\|_{L^m(Q_1)} \le M$, for some $M > 0$ and $m > 5$. Then there exists $\epsilon_1 = \epsilon_1(m, M) > 0$ with the following properties: if*

$Q_1 =]-1+t_0, t_0] \times B_1(0)$

$$\left(\fint_{Q_1} |V|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3} \le \epsilon_1,$$

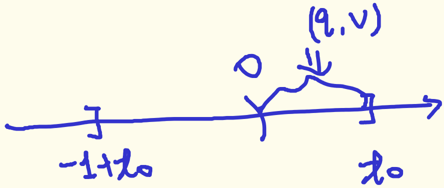
then V is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

à faire

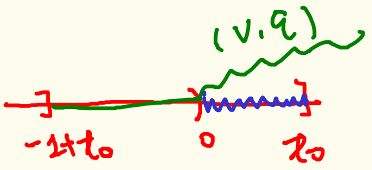
$$\|V\|_{C^{\alpha}_{\mathrm{par}}(Q_{1/2})} \le C(m, \epsilon_1, M) = C(m, M). \tag{2.22}$$

P13

Extension :

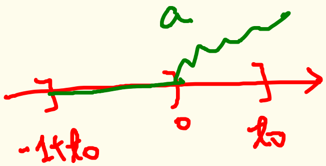


on a des estimations pour (v, q) sur $[0, t_0]$ mais pour appliquer Thm III, il faut ajouter des valeurs sur l'interval $]-1+t_0, 0]$.



$$V = \begin{cases} 0 & (-1+t_0, 0) \\ v & (0, t_0) \end{cases}$$

$$q = \{ \cdots$$



$$a = \begin{cases} 0 & \cdots \\ a & \cdots \end{cases}$$

(Eq-perturbation) : $\partial_t V \cdots + a \cdot \nabla V + \operatorname{div} \cdots + \nabla q_{\tilde{0}}$

si on prend $t_0 << 1$,

$$\left(\int_{-1+t_0}^{t_0} \int_{B_1(0)} |V|^3 \right)^{1/3} + \left(\int_{-1+t_0}^{t_0} \int_{B} |q|^{3/2} \right)^{2/3} \leq C(t_0) < \varepsilon .$$

$\Rightarrow$ V est Hölder continu ds $B_{\frac{1}{2}}(0) \times (-\frac{1}{2}+t_0, t_0)$ $Q_{\frac{1}{2}}$

Theorem ~~3.1~~ III *Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u_0|^2(x)dx \leq \alpha < \infty$. Suppose u_0 is in $L^m(B_2(0))$ with $\|u_0\|_{L^m(B_2(0))} \leq M < \infty$ and $m > 3$. Let us decompose*[1] *$u_0 = u_0^1 + u_0^2$ with $\operatorname{div} u_0^1 = 0$, $u_0^1|_{B_{4/3}} = u_0$, $\operatorname{supp} u_0^1 \Subset B_2(0)$ and $\|u_0^1\|_{L^m(R^3)} \leq C(M, m)$. Let a be the locally in time defined mild solution to Navier-Stokes equations with initial data u_0^1. Then there exists a positive $T = T(\alpha, m, M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies: $u - a \in C^{\gamma}_{\text{par}}(\overline{B_{1/2}} \times [0, T])$, and $\|\underbrace{u-a}_{V}\|_{C^{\gamma}_{\text{par}}(\overline{B_{1/2}} \times [0,T])} \leq C(M, m, \alpha)$, for some $\gamma = \gamma(m) \in (0, 1)$.*

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Theorem 3.2 (Local Hölder regularity of Leray solutions) *Let* $u_0 \in L^2_{loc}(R^3)$ *with* $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u|^2(x)dx \le \alpha < \infty$. *Suppose* u_0 *is in* $C^\gamma(B_2(0))$ *with* $\|u_0\|_{C^\gamma(B_2(0))} \le M < \infty$. *Then there exists a positive* $T = T(\alpha, \gamma, M) > 0$, *such that any Leray solution* $u \in \mathcal{N}(u_0)$ *satisfies*:

$$u \in C^\gamma_{\text{par}}\left(\overline{B_{1/4}} \times [0, T]\right), \quad \textit{and} \quad \|u\|_{C^\gamma_{\text{par}}(\overline{B_{1/4}} \times [0,T])} \le C(M, \alpha, \gamma). \quad (3.12)$$

Pf: $u_0 = u_0^1 + u_0^2$,

$\operatorname{div} u_0^1 = 0$

$\operatorname{supp}(u_0^1) \subseteq B_2(0)$

$\|u_0^1\|_{C^\gamma(B_2(0))} \le M < \infty$.